Dynamic inefficiency with a decreasing returns technology for firms*

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Introduction

Since Diamond's (1965) seminal contribution, several authors have elaborated on the possibility of dynamic inefficiency in various alternative frameworks.

The present paper follows the line of research originated by Tirole (1985).

That contribution studies, in an otherwise standard version of Diamond's model, not only the possibility of bubbles but also the implications of an asset that brings a real rent. Tirole shows that, if rents per period are capitalised in advance and if they increase at the (asymptotic) rate of economic growth, a perfect foresight equilibrium must be efficient. Were this not true, the rent per period would grow at a rate exceeding the rate of interest and its market value would be infinite.

Tirole ((1985) p. 1074) provides several examples of assets bringing a real rent, such as natural resources, land, decreasing return to scale technologies and paintings and jewels for their consumption value.

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* I thank Neil Rankin and Philippe Weil for extremely helpful suggestions on a previous version of this work. I benefited also from the detailed comments of two referees. The present paper was partly written during a stay at Princeton University, whose hospitality is gratefully acknowledged. Usual disclaimers apply. Financial support from C.N.R. and M.U.R.S.T. has been greatly appreciated.

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Since in exogenous growth models dynamically inefficient steady state equilibria prompt for Pareto improving policy actions[1], it is not surprising that each of these examples has been analysed in the literature.

The role of paintings and jewels is outlined by Tirole himself ((1985) p. 1080). An asset of this type provides rents that cannot be capitalised before the creation of the asset itself. In such case, bubbles (and dynamic inefficiency) need not be inconsistent with rents per period growing at the same rate of the economic system, since the flow of rents stemming from a single specific asset does not grow and must be capitalised using the interest rate.

In contrast to this analysis, McCallum (1987) and Homburg (1991), assign to land an explicit role in aggregate production and rule out inefficient equilibria. In fact, in their models, the marginal product of land (the rent) grows at the asymptotical growth rate of output[2]. Clearly, this result depends on the characteristics of the aggregate production function; it does not hold, as suggested by O'Connel and Zeldes ((1988) pp. 441-2, fn. 19), whenever the land income share vanishes in the long run. Homburg (1991) and Rhee (1991) provide two different formalisations for this latter point. Moreover, Rhee notes that the decline in the US land income share, during the post-war period, has not been quick enough to be conclusive.

The case of a decreasing returns-to-scale technology has been analysed by Dechert and Yamamoto (1992) in a (stochastic) overlapping generations model with no population growth[3]. In their setting, rents are distributed to shareholders as dividends and these agents, who are old, sell (non bubbly) stocks to the young. Since the equities values would approach infinity as the interest rate gets closer and closer to nought, Dechert and Yamamoto conclude that "the stock market serves the same purpose as the transversality condition in an infinite-horizon growth model" ((1992) p. 399). This model suffers from two drawbacks: the number of firms is not endogenously determined and, in case of population growth, the share of rents on output would continuously grow over time, a rather counterfactual implication.

In this paper, we study the role of pure profits taking a route that is similar to the one followed by Dechert and Yamamoto. We describe an environment where many identical firms are operative, their number being large enough that perfect competition can be assumed. In our model, the (gross) production function of each firm is strictly convex, i.e. returns to scale are decreasing. Entry is not constrained, so that the number of firms

1 With endogenous growth, the picture changes dramatically: a reduction in the capital stock cannot make everybody better off since it is harmful for growth (see King and Ferguson (1993)). The interest rate may still be lower than the growth rate, giving room to bubbles, but these, in general, are harmful to society (Grossman and Yanagawa (1993)). However, in this paper we do not wish to be involved with the implications of the endogenous growth models.

2 A similar point has been made by Muller and Woodford ((1988) p. 962) while considering a model where finitely and infinitely lived agents coexist.

3 Although Dechert and Yamamoto's model is stochastic, their point concerning dynamic efficiency can be easily recasted in a deterministic framework.

increases over time in response to population growth (and aggregate returns to scale are constant). The presence of a fixed cost, that is to be paid upon entering the market, determines endogenously the number of firms that are operative at every instant of time[4]. We also introduce a constant and exogenous probability of failure affecting firms.

In a context where the entrance of new firms plays an important role, it seemed natural to avoid the rigid timing entailed by the standard two periods overlapping generations model. Hence, to model the "consumers' side", we adopt Buiter's (1988) continuous time framework, which allows both for a positive probability of death at the individual level and for agents disconnectedness in the sense of Weil (1989). However, one can show that, under appropriate conditions (a time-separable utility function and a sufficiently low labour income during the second period of life), our results hold true also in a standard framework *à la* Diamond.

The central result of the present paper is that dynamic inefficiency is impossible only if population does not grow and if the probability of failure for firms is zero. Hence, the presence of a decreasing returns technology distinguishes the present settings from the standard model, where dynamic inefficiency with stationary population is possible (for a direct comparison, see Blanchard (1985)); however, we show that decreasing returns at the firm's level are not *per se* sufficient to solve the problem.

The fixed entry cost proves to be fundamental for our result, since it implies the equality, in the long run equilibrium, between the growth rate for the number of firms and the one for population and output. Therefore, any existing firm does not grow over time and, in discounting future profits, it does not modify the interest rate with the growth rate: if population increases over time the interest rate might be lower than the growth rate.

Dynamic inefficiency is possible also when population is stationary but the probability of failure is positive. The interpretation for this result is again simple: in the steady state, the representative firm does not grow due to the characteristics of the sunk cost; however the firm takes account of the probability of failure in evaluating future profits. This heavier discounting keeps the value of firms positive even if the marginal productivity of capital becomes negative.

In what follows, we start summarising the behaviour of our continuum of agents; we then present the supply-side of the economic system (section II), and we characterise and discuss the steady state properties of the model (section III and IV, respectively). For simplicity, we assume away, throughout the paper, government debt, public expenditure and hence taxation. Section V concludes.

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4 We have in mind a situation where decreasing returns are due, e.g., to the need, for each firm, of purchasing an (indivisible) building or machinery of relevant size. Also, the requirement for entrepreneurs or managers may play a role. Notice that an alternative set of assumptions contemplates the presence of identical firms that compete "*à la* Bertrand": the undercutting behaviour implies that the price is equal to the marginal cost. With decreasing returns to scale, this hypothesis involves the presence of some profit, which, as will be clear in the sequel, is important to determine endogenously the number of firms.

1 Aggregate consumption and asset accumulation

Following Blanchard (1985), we assume that each individual agent faces a constant instantaneous probability of death, λ, that also represents, due to the law of large numbers, the fraction of each cohort that dies at every instant. This hypothesis, together with the one of a constant birth rate β, as in Buiter (1988), has the relevant merit of allowing aggregation.

1.1 Individual consumption

To keep the analysis simple, we adopt, at the single agent's level, a logarithmic specification for the time separable utility function. Thus, the representative individual born at time s maximises, at time t:

$$U(t,s) = \int_t^\infty ln[c(\tau,s)]e^{-(\theta+\lambda)(\tau-t)}d\tau$$

s.t. $\dot{a}(t,s) = [r(t)+\lambda]a(t,s) + w(t,s) - c(t,s)$
where $c(t,s)$ is consumption, $a(t,s)$ the stock of assets and $w(t,s)$ labour income, all considered at time t for the individuals born at time s; θ and $r(t)$ are, respectively, the intertemporal time preference rate and the interest rate. A dot over a variable denotes, as usual, its derivative with respect to time. Notice that the usual Blanchard-Yaari actuarially fair insurance mechanism is at work. As in Blanchard (1985), any surviving agent obtains from the insurance companies the fraction λ of her assets $a(t,s)$ at every instant of time; her wealth goes to the insurer when she dies[5]. The following "no-Ponzi game" condition holds:

$$\lim_{\tau\to\infty} = a(\tau,s)\exp\left(-\int_t^\tau [r(z)+\lambda]dz\right) = 0$$

We assume, as in Blanchard, ((1985) p. 235), that the effect of retirement can be stylised by letting the labour income decline with age at a constant rate ρ. With logarithmic preferences, a necessary condition for dynamic inefficiency in Blanchard's model is a sufficiently high ρ, namely, $\rho > \theta$[6].

Following usual methods, it is possible to show that the consumption behaviour, at the individual level, is described by the following equation:

$$c(t,s) = (\theta+\lambda)[a(t,s)+h(t,s)]$$

5 See also Blanchard and Fischer (1989). Notice that the presence of this mechanism is due to the absence of any bequest motive.

6 With isoelastic preferences, the necessary condition becomes $(1-S)\lambda+\rho > S\theta$, where S is the elasticity of intertemporal substitution. Hence, the lower S, the looser becomes this condition.

where $h(t,s)$, human wealth, is defined as:

$$h(t,s) = \int_t^{\infty} w(\tau,s) \exp\left(-\int_t^{\tau} [r(z)+\lambda]dz\right) d\tau$$

1.2 Population dynamics and aggregation

Given our assumptions concerning death and birth rates, the population at time t is:

$$N(t) = N(0)e^{nt}; \tag{1}$$

with no loss of generality, we set $N(0) = 1; n = \beta - \lambda$ is the constant population growth rate. In this model, a positive population growth rate may be interpreted as an effect of immigration (see Barro and Sala-i-Martin, (1995) ch. 9).

Since logarithmic preferences entail, at the individual level, a consumption function that is linear in total wealth, it is straightforward to obtain its aggregate counterpart[7]:

$$C = (\theta+\lambda)(A+H) \tag{2}$$

Notice that, from equation (2) on, we take as understood the time index t whenever this is not confusing. Exploiting the following boundary condition:

$$\lim_{\tau\to\infty} H(\tau) \exp\left(-\int_t^{\tau} [r(z)+\lambda]dz\right) = 0$$

we obtain, by means of standard techniques, the differential equations for assets and human wealth.

$$\dot{A} = rA + W - C \tag{3}$$

$$\dot{H} = (r+\beta+\rho)H - W \tag{4}$$

The absence of the λA term in equation (3) is due to the insurance companies' activity, which transfer resources from those who die to those who survive: this process is not affected by the birth rate. The βH term in equation (4) reflects the fact that all the agents, even the newborn ones, have the same life expectancy (Buiter (1988) p. 283). As intuition suggests, the discount rate of human wealth is enhanced by ρ, the rate of decline of labour income.

7 The population aggregate corresponding to any individual stock or flow variable $x(t,s)$ is defined as:

$$X(t) = \int_{-\infty}^{t} x(t,s)\beta e^{\beta s} e^{-\lambda t} ds$$

Differentiating with respect to time equation (2) and exploiting equations (3) and (4), we get the law of motion for consumption:

$$\dot{C} = (r + n + \rho - \theta)C - (\theta + \lambda)(\beta + \rho)A \tag{5}$$

It is useful to compare this equation with the corresponding one in the Ramsey model with population growth (see e.g. Blanchard and Fischer (1989)). In that framework, the second addendum on the right hand side is missing and per capita consumption is multiplied by $(r - \theta)$. The presence of the n term in our equation reflects the fact that existing agents are not concerned about the unborn people[8].

Equations (3) and (5) summarise the aggregate behaviour of our continuum of disconnected families.

2 A decreasing returns-to-scale technology

In what follows, we assume that the production side of the economy is composed of v firms, each producing y_i units of the final good, which is used both for consumption and investment[9]. Every firm operates with a decreasing returns technology; capital and labour are the arguments of the gross production function:

$$y_i = F(K_i, L_i)$$

In order to be able to obtain a closed form steady state, we assume that the production function is homogeneous of degree μ, and that all the existing firms are identical. Hence, we may express the gross production function in intensive form:

$$y_i = F(k, 1)L_i^{\mu} = f(k)L_i^{\mu} \tag{6}$$

where k is the capital/labour ratio. Since, by hypothesis, all firms are identical, k represents also the aggregate capital/output ratio. We assume $f(k)$ to be strictly concave and to satisfy the condition: $\lim_{k\to\infty} f_k(k) = 0$. We also assume that capital depreciates at the constant rate δ; hence, due to profit maximisation, the real interest rate r is equal to the net marginal productivity of capital:

$$r = F_{K_i}(K_i, L_i) - \delta = f_k(k)L_i^{\mu-1} - \delta \tag{7}$$

Our assumption about the degree of homogeneity of the production function implies that "pure profits" (rents) are a constant share $(1 - \mu)$ of gross output:

$$\pi_i = y_i - \delta K_i - rK_i - wL_i = (1 - \mu)f(k)L_i^{\mu} \tag{8}$$

8 To get an intuition for this point, consider that, at the per capita level, consumption is not affected by population growth.

9 We continue to take the time index t as understood.

where π_i and w denote, respectively, rents and wages (at time t). Notice also that gross aggregate output is :

$$Y = vy_i = vf(k)L_i^{\mu} = f(k)L^{\mu}v^{1-\mu}$$

where the last equality exploits symmetry among firms. Aggregate output is homogeneous of degree one in capital, labour and in the number of firms.

The presence of rents implies that the ownership of a generic firm has a fundamental value, Q_i, which is given by :

$$Q_i(t) = \int_t^{\infty} \pi_i(\tau) \exp\left(-\int_t^{\tau} [r(\zeta) + p]d\zeta\right) d\tau \tag{9}$$

In equation (9), we have introduced the assumption of a constant and exogenous probability of failure affecting firms. The lifetime of firms is exponentially distributed; hence $\exp[-p(\tau - t)]$ is the probability of being operative at time τ for a firm existent at time t. Since firms are owned by risk averse consumers, the use of the interest rate in the discount integral may seem inadequate. However, we assume that the number of existing firms is large (which is consistent with our perfectly competitive framework) and that consumers perfectly diversify their portfolios. Therefore each individual's return is never affected by the failure of a specific firm; rather, consumers perceive p as the (non stochastic) share of equities in their portfolio that is "vanishing" at each instant of time. Hence, every investor behaves as if she were risk neutral and firms use the interest rate to discount profits.

Differentiation of the forward-looking equation (9) yields the familiar arbitrage equation :

$$\dot{Q}_i(t) = -\pi_i(t) + [r(t) + p]Q_i(t) \tag{10}$$

implying that expected capital gains $\dot{Q}_i(t) - pQ_i(t)$ plus profits $\pi_i(t)$ must equal what could be obtained investing the value of the firm on the bond market, $r(t)Q_i(t)$.

We now assume that entry of new firms in the market is unrestricted but costly. More specifically, we assume that every firm, before starting production, has to bear a strictly positive fixed cost ϕ, denominated in output.

At the level of stylised facts, a cost with these characteristics could be entailed by the need, for the entrant firms, of purchasing a building, a plant and/or some machinery. In this case, the probability of failure p stylises the presence of a technology shock causing the breakdown of the firm equipment[10]. Alternatively, we may think that every firm must be run by an entrepreneur and that she must bear some cost to train herself before

[10] A fixed scrap value for the firm can be embodied into the model without affecting the qualitative results.

starting to manage the firm[11]. The free entry assumption implies that the value of every firm must be equal to the fixed cost, $Q_i = \phi$. Therefore, the differential equation (10) reduces to the following ordinary one:

$$(r+p)\phi = \pi_i$$

which provides a relation between per capita capital and per capita number of firms. To see this, substitute for r and π_i using equations (7) and (8) and define $x = v/L = (L_i)^{-1}$, to get:

$$[f_k(k)x^{1-\mu} - \delta + p]\phi = (1-\mu)f(k)x^{-\mu} \tag{11}$$

This equation may be conveniently seen as a function:

$$x = g(k) \tag{12}$$

Moreover, differentiation of (11) shows that $g_k(k) > 0$, implying that the accumulation of capital increases the number of existing firms.

Notice that the presence of p is made possible by the assumption of free entry. A similar hypothesis would not be possible in the land model: since land is conceived as a non-reproducible factor, its disruption would lead the economic system to the collapse.

3 Characterisation of the steady-state equilibrium

This section identifies the "long-run" properties of our economy. We set up, first, a dynamical system involving the three variables of the model: per capita consumption, capital and number of firms. We then characterise the steady state equilibrium: we argue that the long-run equilibrium of the model may be dynamically inefficient and we show that, under a mild condition, the steady state is unique and saddlepath stable. Throughout the section, stars will denote steady state values for the variables.

Since we have assumed away public debt, assets are given by the sum of capital and the value of firms; hence, the aggregate consumption equation (5) is reformulated in per capita terms as follows:

$$\dot{c} = (r + \rho - \theta)c - (\theta + \lambda)(\beta + \rho)(k + \phi x) \tag{13}$$

11 This interpretation requires that the presence of entrepreneurs does not create moral hazard problems to the possibility of portfolio diversification. Also, the reward for entrepreneurs' efforts should be deducted from profits in equation (9). (The qualitative results of the model are robust to a specification where the reward to entrepreneurs is proportional to the wage bill, as long as pure profits remain strictly positive. The reward could as well be a fraction of rents at every instant of time.) Notice also that, according to this interpretation, p should be equal to the probability of death; moreover, entrepreneurs should bear the one-off cost at the instant of birth.

where ϕx stands for the per capita value of firms. Since per capita income may be used as an addition to per capita capital, for consumption or to increase the number of firms, we can write:

$$\dot{k} = f(k)x^{1-\mu} - \delta k - c - \phi(\dot{x} + px) - n(k + \phi x) \tag{14}$$

The relation (12) between x and k closes the model.

It will prove useful to reformulate the system composed of equation (13) and (14). Exploiting equations (7) and (12) we get:

$$\dot{c} = [f_k(k)g(k)^{1-\mu} - \delta + \rho - \theta]c - (\theta + \lambda)(\beta + \rho)[k + \phi g(k)] \tag{13'}$$

To obtain the dynamic equation for capital, differentiate with respect to time equation (12), substitute out for $\dot{x}$ in equation (14) and solve the resulting expression for $\dot{k}$:

$$\dot{k} = [1 + \phi g_k(k)]^{-1}\{f(k)g(k)^{1-\mu} - \delta k - c - \phi p g(k) - n[k + \phi g(k)]\} \tag{14'}$$

Notice that, since $\dot{x} = g_k(k)\dot{k}$, in steady state the aggregate rents ($= (r+p)\phi x L$) grows at the aggregate growth rate (n), an important difference with Tirole's (1985) paper.

We now show that the condition for dynamic efficiency is the same that holds in the standard model with constant-returns-to-scale at the firm level; we will then characterise the steady state.

Proposition 1. *The steady state is dynamically efficient if* $r \geqslant n$.

Proof. Differentiating with respect to k the steady state locus for capital, obtained from equation (14'), one gets:

$$\left.\frac{\partial c}{\partial k}\right|_{\dot{k}=0} = f_k(k)g(k)^{1-\mu} - \delta + (1-\mu)f(k)g_k(k)g(k)^{-\mu} - \phi p g_k(k) - n[1 + \phi g_k(k)]$$

Using (11) to substitute out for $(1-\mu)f(k)g(k)^{-\mu}$ in the last equation, it is immediate to realise that consumption is maximised when the interest rate is equal to the population growth rate:

$$\left.\frac{\partial c}{\partial k}\right|_{\dot{k}=0} = [f_k(k)g(k)^{1-\mu} - \delta - n][1 + \phi g_k(k)] = (r - n)[1 + \phi g_k(k)] \tag{15}$$

Hence, per capita consumption is decreasing when $r < n$. □

We now prove:

Proposition 2. *If a non-trivial steady state exists, the interest rate* (r^*) *must be i) lower than* $\beta + \theta$, *and ii) higher than* $\max\{-p, \theta - \rho\}$.

Proof. Part *i)* follows Blanchard ((1985) pp. 237-38) and is proved by contradiction. If $r^* \geqslant \beta + \theta$, since, from (13'),

$$(r^* + \rho - \theta)c^* = (\theta + \lambda)(\beta + \rho)[k^* + \phi g(k^*)],$$

it follows that

$$(\beta + \rho)c^* \leqslant (\theta + \lambda)(\beta + \rho)[k^* + \phi g(k^*)],$$

or

$$c^* \leqslant (\theta + \lambda)[k^* + \phi g(k^*)]$$

However, from equation (14'), we obtain that, in steady state,

$$c^* = f(k^*)g(k^*)^{1-\mu} - \delta k - \phi p g(k^*) - n[k^* + \phi g(k^*)]$$

Therefore

$$(\theta + \lambda)[k^* + \phi g(k^*)] \geqslant f(k^*)g(k^*)^{1-\mu} - \delta k^* - \phi p g(k^*) - n[k^* + \phi g(k^*)]$$

or

$$r^*[k^* + \phi g(k^*)] \geqslant (\theta + \beta)[k^* + \phi g(k^*)] \geqslant f(k^*)g(k^*)^{1-\mu} - \delta k^* - \phi p g(k^*)$$

This chain of (weak) inequalities implies that

$$r^* k^* + [(r^* + p)\phi g(k^*)] \geqslant f(k^*)g(k^*)^{1-\mu} - \delta k^*$$

Multiply both sides of (11) by $g(k^*)$ and subtract the resulting equation, side by side, from the last expression to get

$$r^* k^* \geqslant \mu f(k^*)g(k^*)^{1-\mu} - \delta k^*$$

or, using (7),

$$f_k(k^*)g(k^*)^{1-\mu}k^* \geqslant \mu f(k^*)g(k^*)^{1-\mu}$$

that is the desired contradiction. □

To prove part *ii)* one notices that equation (11) requires that r^* is higher than $-p$; r^* must also be higher than $\theta - \rho$ (equation 13') so as to allow for a positive steady state consumption. □

The following Assumption 1 : $\frac{f_{kk}(k)f(k)}{f_k(k)^2} \leqslant \frac{\mu-1}{\mu}$ allows us to prove a lemma useful to show existence and uniqueness of the steady state equilibrium.

Lemma. *If Assumption 1 holds, then* $\left(\frac{f_{kk}(k)}{f_k(k)} + (1-\mu)\frac{g_k(k)}{g(k)}\right) < 0.$

Remark. Assumption 1 essentially requires that the production function be such that, at the single firm level, wages increase with the capital/labour ratio[12].

Proof. By totally differentiating equation (11) we obtain:

$$\frac{g_k(k)}{g(k)} = \frac{(1-\mu)f_k(k) - \phi f_{kk}(k)g(k)}{(1-\mu)[\phi f_k(k)g(k) + \mu f(k)]}$$

The introduction of this ratio into $\left(\frac{f_{kk}(k)}{f_k(k)} + (1-\mu)\frac{g_k(k)}{g(k)}\right)$, gives:

$$\left(\frac{f_{kk}(k)}{f_k(k)} + \frac{(1-\mu)f_k(k) - \phi f_{kk}(k)g(k)}{\phi f_k(k)g(k) + \mu f(k)}\right) =$$

$$\frac{\mu f_k(k)}{\phi f_k(k)g(k) + \mu f(k)}\left(\frac{f_{kk}(k)f(k)}{f_k(k)^2} + \frac{(1-\mu)}{\mu}\right)$$

which is negative if Assumption 1 holds. □

The results obtained in *Proposition* 2 and in the *Lemma* are useful to prove:

Proposition 3. *If Assumption 1 holds, the steady state equilibrium of the dynamic system composed of equations (13'-14') : i) exists and is unique; ii) is locally saddlepath stable.*

Proof. Part *i)*. To prove existence and uniqueness, notice first that:

$$\begin{aligned}\left.\frac{\partial c}{\partial k}\right|_{\dot c = 0} =&(\theta+\lambda)(\beta+\rho)\left[\frac{[1+\phi g_k(k)]}{r+\rho-\theta}\right.\\ &\left.-[f_{kk}(k)g(k)^{1-\mu} + (1-\mu)f_k(k)g(k)^{-\mu}g_k(k)]\frac{[k+\phi g(k)]}{(r+\rho-\theta)^2}\right]\\ =&(\theta+\lambda)(\beta+\rho)\left[\frac{[1+\phi g_k(k)]}{r+\rho-\theta}\right.\\ &\left.-f_k(k)g(k)^{1-\mu}\left(\frac{f_{kk}(k)}{f_k(k)} + (1-\mu)\frac{g_k(k)}{g(k)}\right)\frac{[k+\phi g(k)]}{(r+\rho-\theta)^2}\right]\end{aligned}$$

If Assumption 1 holds, by the lemma we see that the term in the big round brackets is negative. Hence:

$$\left.\frac{\partial c}{\partial k}\right|_{\dot c = 0} > (\theta+\lambda)(\beta+\rho)\left[\frac{[1+\phi g_k(k)]}{r+\rho-\theta}\right]$$

12 Recall that the wage is given by $w = \frac{1}{L_i}[\mu f(k)L_i^{\mu} - f_k(k)L_i^{\mu-1}K_i] = L_i^{\mu-1}[\mu f(k) - f_k(k)k]$. If, for a given L_i, $\frac{\partial w}{\partial k} > 0$, Assumption 1 is satisfied. To see this, notice that $\frac{\partial w}{\partial k} > 0$ implies $\frac{f_{kk}(k)k}{f_k(k)} < \mu - 1$. The fact that $\mu f(k) > f_k(k)k$ involves, in its turn, that Assumption 1 holds.

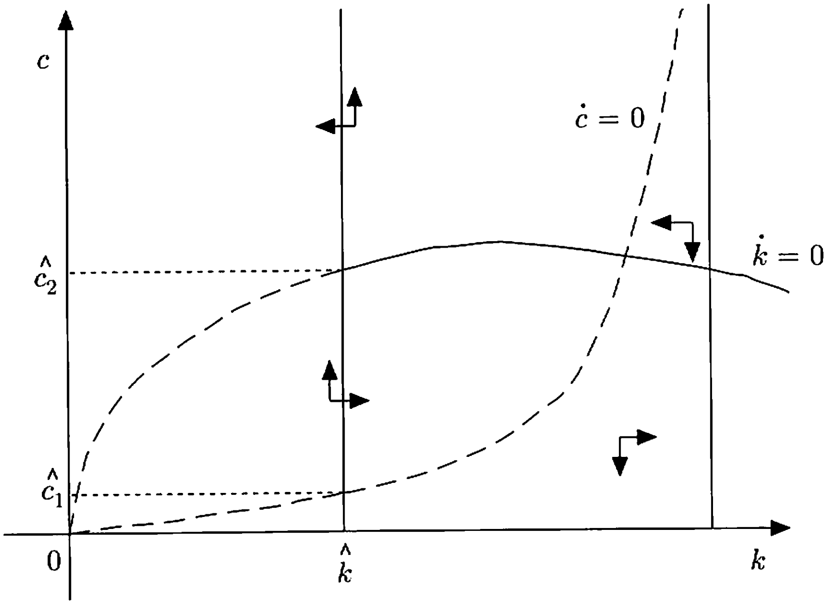

Figure 1 : *Qualitative dynamics of per capita consumption and capital*

Recalling $\frac{\partial c}{\partial k}\big|_{\dot{k}=0}$ from (15), we notice that $\frac{\partial c}{\partial k}\big|_{\dot{c}=0} > \frac{\partial c}{\partial k}\big|_{\dot{k}=0}$ when :

$$(\theta + \lambda)(\beta + \rho)[1 + \phi g_k(k)] > (r - n)[1 + \phi g_k(k)](r + \rho - \theta)$$

which is satisfied for $r^* \in (-(\lambda + \rho), \beta + \theta)$. From Proposition 2, we know that, if a meaningful equilibrium exists, then

$$r^* \in (\max\{-p, \theta - \rho\}, \beta + \theta)$$

Hence, in the relevant interval for r^*, the consumption locus is always steeper than the capital locus and, if a non-trivial steady state equilibrium exists, it is unique.

To prove existence, consider $\hat{k}$, the capital level such that $\hat{r} = \beta + \theta$. Let $\hat{c}_1$ be the consumption level obtained by setting $\dot{c} = 0$ when $k = \hat{k}$. Simple calculations show that $\hat{c}_1 = (\theta + \lambda)[\hat{k} + \phi g(\hat{k})]$. Let $\hat{c}_2$ be the consumption level obtained by setting $\dot{k} = 0$ when $k = \hat{k}$. From equation (14') one immediately obtains :

$$\hat{c}_2 = f(\hat{k})g(\hat{k})^{1-\mu} - \delta\hat{k} - \phi p g(\hat{k}) - n[\hat{k} + \phi g(\hat{k})]$$

By properly adapting the reasoning in Proposition 2, one gets that $\hat{c}_2 > \hat{c}_1$. Now notice, form equation (10) and from equation (13) that, when r tends to $\max\{-p, \theta - \rho\}$, the $\dot{c} = 0$ locus has a vertical asymptote. Hence, a non-trivial steady state equilibrium exists. (Refer to Figure 1). □

To prove part *ii)*, notice that the linearisation of the dynamic system (13'-14') in the neighbourhood of the steady state gives:

$$\begin{bmatrix} r^* + \rho - \theta & \begin{array}{c} g(k^*)^{-\mu}[f_{kk}(k^*)g(k^*) + (1-\mu)f_k(k^*)g_k(k^*)]c^* \\ -(\theta+\lambda)(\beta+\rho)[1+\phi g_k(k^*)] \end{array} \\ -1 & (r^* - n)[1+\phi g_k(k^*)] \end{bmatrix}$$

The determinant of the matrix above can be expressed as:

$$Det = [(r^* + \rho - \theta)(r^* - n) - (\theta + \lambda)(\beta + \rho)][1 + \phi g_k(k^*)]$$

$$+ c^* f_k(k^*)g(k^*)^{1-\mu}\left(\frac{f_{kk}(k^*)}{f_k(k^*)} + (1-\mu)\frac{g_k(k^*)}{g(k^*)}\right)$$

Proposition 2 grants that, in the steady state, the first addendum of the determinant is negative; Assumption 1 and the lemma imply the negativity of the second addendum. □

The above results allow us to draw the following phase diagram, describing qualitatively the dynamics of system (13'-14'). For $k < \hat{k}$ the loci $\dot{c} = 0$ and $\dot{k} = 0$ are dashed, since their properties have not been determined analytically[13].

4 Discussion

Proposition 1 simply tells us that, as in the standard constant-return-to-scale model, aggregate per capita consumption is maximised when the marginal productivity of capital is equal to the population growth rate (and the economy is in *Golden Rule*).

The chain of (weak) inequalities obtained to prove part *i)* of Proposition 2 has a precise economic meaning. It implies that the fraction μ of per capita output is lower than (or equal to) per capita capital income, a clear impossibility given our hypothesis concerning the degree of homogeneity of the production function.

Proposition 2 tells us also that, in our set-up, the steady state needs not be dynamically efficient, i.e. r^* may be lower than n. To interpret this result, notice, first, that a suboptimal outcome is possible also when firms do not face a probability of failure. In this case, the marginal productivity of capital must be positive, but it need not be higher than the population growth rate. The economic intuition for this result is simple: the structure of the entry cost implies that, in the steady state, the growth rate of the

[13] To sign the "arrow fields", compute $\left.\frac{\partial \dot{k}}{\partial c}\right|_{\dot{k}=0}$ and $\left.\frac{\partial \dot{c}}{\partial c}\right|_{\dot{k}=0}$.

number of firms is equal to the one for output. Hence the existing firms do not grow over time and, when discounting future profits, they do not modify the interest rate with the growth rate. The fact that, in our perfect foresight framework, rational agents can capitalise in advance the value of still non-existing (or simply non-producing) firms is not relevant, since the value of these firms, before they sank the cost, is nought. Hence, the economic system behaves as if it could not capitalise in advance future rents, as in Tirole's example of paintings; the fact that, in our framework, aggregate rents are increasing over time becomes irrelevant.

Dynamic inefficiency is possible also when there is no population growth but the probability of failure is positive. The interpretation is, again, simple. The constancy of the entry cost implies again that, in the steady state, the number of firms does not grow over time. Since the long run output is constant, due to the absence of population growth, the representative firm does not grow; however it takes account of the probability of failure in evaluating future profits. This heavier discounting keeps the value of firms positive even if the marginal productivity of capital becomes negative. This result contrasts with the one by Dechert and Yamamoto, who suggested that the presence of aggregate decreasing returns to scale resolves into a non negative interest rate: in our model this would be true only if firms had infinite horizon.

Our analysis implies that dynamic efficiency may fail, but does not establish precise conditions for this to happen. To provide an approximate idea about the parameters values involving dynamic inefficiency, we simulate simple economic systems by means of a Gauss program. We choose a Cobb-Douglas production function at the single firm level (hence, firm's net output is given by $AK_i^{\alpha\mu}L_i^{(1-\alpha)\mu} - \delta K_i$) and we fix $\mu = 0.9$ and $\alpha = 0.26$ so that the labour share of income is about two thirds. The level for μ might seem high; however lower values imply that the portion of the parameters space involving dynamic inefficiency is larger. The capital depreciation parameter is assumed equal to 10% while we set A to 10. This choice, together with $\phi = 0.1$, implies the existence of a large number of firms not only in the long run but also during (almost) the entire transition towards the steady state; however the values for A and ϕ proved not to affect the long-run interest rate.

The first result concerns the version of the model based on the logarithmic utility function that has been used throughout the paper. We fixed the intertemporal subjective discount rate to 2%, the probability of death to 1.5% and the birth rate to 2.5%, implying that the population growth (or migration) rate is a modest 1%. We assumed that the labour income is declining at a rate equal to 2%. We then let our program to find the "probability of failure" for firms that implies dynamic inefficiency.

For this set of parameters, the Gauss routine determines that the steady state equilibrium is dynamically inefficient if $p \geqslant 4.884\%$, a high but still reasonable value.

We then consider a version of the model allowing for the isoelastic utility function. Hence, the representative individual born at time s maximises, at time t:

$$U(t,s)=\int_t^{\infty}[c(\tau,s)^{(1-R)}]/(1-R)e^{-(\theta+\lambda)(\tau-t)}d\tau,$$

where R is the inverse of the elasticity of intertemporal substitution. Under this assumption, we added $p = 2\%$ to the set of numerical values provided for the parameters, and we let our program to determine the highest possible level for the elasticity of intertemporal substitution that implies overaccumulation of capital. It turns out that the steady state equilibrium is inefficient if the elasticity of intertemporal substitution is lower than 0.7129, a value compatible with most of the available estimates[14].

5 Concluding remarks

Tirole, in his well-known (1985) paper, suggested that decreasing returns technologies could be considered as assets bringing a real rent. However, in his analysis, he did not discuss explicitly such a case; rather he assumed that the total rent is constant over time and he concluded that the standard exogenous growth model might lead to a dynamically inefficient equilibrium. Dechert and Yamamoto analysed explicitly a stochastic overlapping generations model with aggregate production characterised by decreasing returns; they found that the stock market is effective in ruling out the possibility of an inefficient steady state.

The perspective of this paper is somewhat different: decreasing returns technologies are regarded as assets that can be "produced" at some cost. This approach allowed us to determine endogenously the number of firms and we could consider population growth without reaching the rather dissatisfying conclusion that the share of rents must steadily grow over time.

We assumed that the entry cost for firms, i.e. the price that must be paid to replicate the asset, is constant over time and we found that a dynamically inefficient long run equilibrium is possible in case of population growth. This result is related to the constancy of the entry cost, which implies that any existing firm does not grow over time. Hence, in discounting future profits, firms do not modify the interest rate with the growth rate: if population increases, the interest rate might be lower than the growth rate.

14 If we consider a lower birth rate, setting $\beta = 0.015$, therefore ruling out population growth, the critical value becomes 0.4861. If the departure from our baseline parameters consists in a lower probability of failure for firms, $p = 0.015$, the model implies dynamic inefficiency for an elasticity of intertemporal substitution lower than 0.6088. The "perturbation" that seems to cause the most relevant reduction in the critical value is a decrease in ρ from 2% to 1%. In this case the growth-corrected interest rate becomes negative when the elasticity of intertemporal substitution is lower than 0.1685.

Notice that, in contrast to Tirole (1985), the growth of the aggregate rent is not relevant for this argument.

Since decreasing returns technologies are considered "reproducible assets" we could introduce a probability of failure for firms. This distinguishes our model also from the ones considering the role of land (Mc Callum (1987), Homburg (1991) and Rhee (1991)). It turns out that these models have very different implications when compared to the framework developed in the present paper : they suggest that, whenever the aggregate rent does not vanish in the long run, inefficient equilibria are ruled out. In our model, if the probability of failure is positive, dynamically inefficient equilibria are possible even with constant aggregate rent and stationary population. As we pointed out earlier, in the steady state, the representative firm does not grow due to the characteristics of the sunk cost; however, the firm takes account of the probability of failure in evaluating future profits. This heavier discounting keeps the value of firms positive even if the marginal productivity of capital becomes negative.

Hence, the present paper clarifies that decreasing returns technologies do not necessarily rule out dynamic inefficiency. Also, some numerical exercises suggest that the parameter space involving capital overaccumulation with decreasing returns at the firm's level is relevant.

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